Caringbah High School

2019 Year 12 Trial HSC Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen.
- Board-approved calculators may be used.
- A reference sheet with standard formulae and integrals is provided.
- In Questions 11–14, show relevant mathematical reasoning and/or calculations.

Total marks – 70

Section I (Pages 3–5)

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II (Pages 6–11)

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1. The polynomial $P(x) = x^4 - kx^3 - 2x + 33$ has $(x - 3)$ as a factor.
   What is the value of k ?

   (A) $-4\frac{4}{9}$
   (B) -4
   (C) 4
   (D) $4\frac{4}{9}$

2. Which is the correct condition for $y = mx + b$ to be a tangent to $x^2 = 4ay$?

   (A) $am^2 + b = 0$
   (B) $am^2 - b = 0$
   (C) $am + b = 0$
   (D) $am - b = 0$

3. The roots of $3x^3 - 2x^2 + x - 1 = 0$ are α, β and γ.
   What is the value of $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$?

   (A) $-\frac{1}{9}$
   (B) $-\frac{2}{9}$
   (C) 1
   (D) $\frac{2}{9}$

4. Evaluate $\lim_{x\to\infty}\left(\frac{1+x}{2-x}\right)$.

(A) -2 (B) -1

(C) 1 (D) 2

5. What is the correct expression for $\int \frac{dx}{9+4x^2}$?

(A) $\frac{1}{4}\tan^{-1}\left(\frac{2x}{3}\right)$ (B) $\frac{1}{3}\tan^{-1}\left(\frac{2x}{3}\right)$

(C) $\frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)$ (D) $\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)$

6. A particle moves in simple harmonic motion so that its velocity, v, is given by $v^2 = 6 - x - x^2$.

Between which two points does it oscillate?

(A) $x = 2$ and $x = -3$ (B) $x = -2$ and $x = 3$

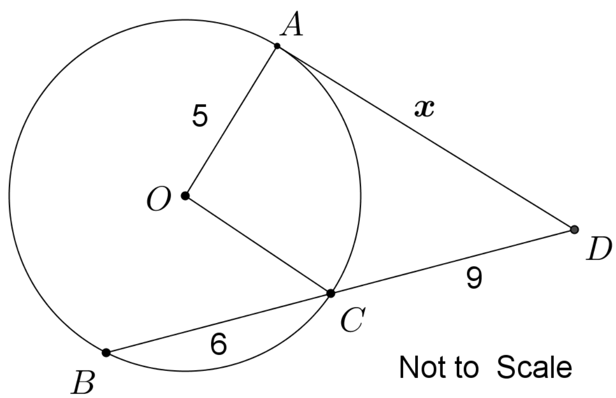
(C) $x = 1$ and $x = 2$ (D) $x = 6$ and $x = 3$

7. $\tan^{-1}(-1) =$

(A) $\frac{-3\pi}{4}$ (B) $\frac{-\pi}{4}$

(C) $\frac{\pi}{4}$ (D) $\frac{3\pi}{64}$

8.



Segment AD lies on a tangent to the circle centre, O, radius 5 cm.

BC is 6 cm and CD is 9 cm.

Find the exact length of AD.

(A) $\sqrt{15}$ (B) $3\sqrt{6}$

(C) $3\sqrt{10}$ (D) $3\sqrt{15}$

9. The general solution for $\cos 2x = -\frac{1}{2}$, where , $n = 0, \pm 1, \pm 2, \ldots$ is

(A) $x = n\pi + (-1)^n \frac{\pi}{3}$ (B) $x = n\pi + (-1)^n \frac{\pi}{6}$

(C) $x = n\pi \pm \frac{\pi}{6}$ (D) $x = n\pi \pm \frac{\pi}{3}$

10. A flat semi-circular disc is being heated so that the rate of increase of the area (A m^2), after t hours, is given by $\frac{dA}{dt} = \frac{1}{4}\pi t$

Initially the disc has a radius of 4 metres.

Which of the following is the correct expression for the area after t hours?

(A) $A = \frac{1}{4}\pi t^2 + 8\pi$ (B) $A = \frac{1}{8}\pi t^2 + 8\pi$

(C) $A = \frac{1}{4}\pi t^2 + 16\pi$ (D) $A = \frac{1}{8}\pi t^2 + 16\pi$

End of Section I

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section .

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Use the Question 11 Writing Booklet. **Marks**

a) Calculate the acute angle between the lines $x - 5y - 2 = 0$ and $x - 2y = 0$ to the nearest degree. **2**

b) Solve the inequality $\dfrac{3x - 2}{x + 1} \geq 5$. **2**

c) i) Express $\cos 2x$ in terms of $\sin^2 x$. **1**

ii) Hence evaluate $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1}{x \sin x}$. **1**

d) Evaluate $\displaystyle\int_0^3 x\sqrt{9 - x^2}\, dx$ using the substitution $u = 9 - x^2$. **3**

e) If A is the point $(-2, -1)$ and B is the point $(1, 5)$, find the coordinates of the point P which divides the interval AB externally in the ratio 2:5. **2**

f) Ms Namvar bought a slurpy in Port Douglas which had a temperature of 5° C. The temperature in Port Douglas was 35° C. The slurpy warms at a rate proportional to the difference between the air temperature and the temperature (T) of the slurpy.

That is, T satisfies the equation $\dfrac{dT}{dt} = k(T - 35)$.

i) Show that $T = 35 + Ae^{kt}$ satisfies this equation. **1**

ii) If the temperature of the slurpy after ten minutes is 10° C, find its temperature, to the nearest whole degree, after 20 minutes. **3**

End of Question 11

Question 12 (15 Marks) Use the Question 12 Writing Booklet. **Marks**

a) The polynomial $P(x)$ is given by $P(x) = x^3 + bx^2 + cx - 10$ where b and c are constants. The three zeroes of $P(x)$ are -1, 2 and α.

i) Find the values of b and c. **2**

ii) Hence or otherwise find the value of α. **1**

b) The equation $2x^3 - 6x + 1 = 0$ has roots α, β and γ. **2**

Evaluate $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$.

c) By considering the expansion for $\tan(\alpha - \beta)$, find x so that **3**

$$\tan^{-1} x = \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{3}\right).$$

d) Consider the function $f(x) = e^{2x} + 6e^{x} + 9$.

i) Explain why $y = f(x)$ has an inverse function $y = f^{-1}(x)$ for all x. **1**

ii) Draw a neat sketch of $y = f(x)$ and $y = f^{-1}(x)$, showing all intercepts and asymptotes. **2**

iii) Find the equation of the inverse function in terms of x. **3**

iv) Hence or otherwise solve $e^{2x} + e^{x} = 6$. **1**

End of Question 12

Question 13 (15 Marks) Use the Question 13 Writing Booklet. **Marks**

a) Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 2x \, dx$. **3**

b) Given the parametric equations in terms of θ, **1**

$x = 3\sin\theta$ and $y = 4\cos\theta$,

find the Cartesian equation.

c) i) Express $3\sin x - 2\cos x$ in the form $R\sin(x - \alpha)$. **2**

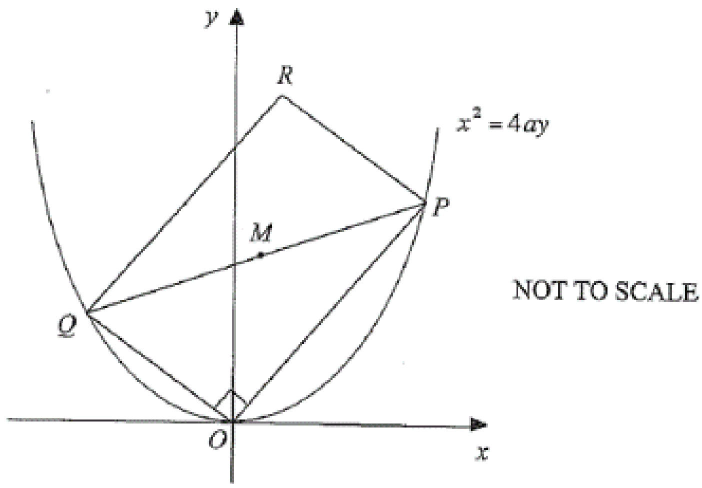
ii) Hence solve $3\sin x - 2\cos x = 1,\ 0 \leq x \leq \frac{\pi}{2}$. **1**

Give your answer correct to 3 significant figures.

Question 13 continued on next page

Question 13 (continued)

d) Two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The chords OP and OQ meet at right angles at the origin. M is the midpoint of the chord PQ. R is a point (not on the parabola) such that $OPRQ$ is a rectangle, as shown in the diagram below.



i) Show that $pq = -4$. **1**

ii) Explain why R has the coordinates $\left(2a(p+q), a(p^2+q^2)\right)$. **1**

iii) Find the equation of the locus of R. **2**

e) A particle oscillates in a straight line under simple harmonic motion. At time, t, it has displacement of x metres from a fixed point O on the line. It's velocity, v ms^{-1}, is given by $v^2 = 32 + 8x - 4x^2$.

i) Find and expression for the particle's acceleration in terms of x. **1**

ii) Find the period and amplitude of the motion. **2**

iii) Find the maximum speed of the particle. **1**

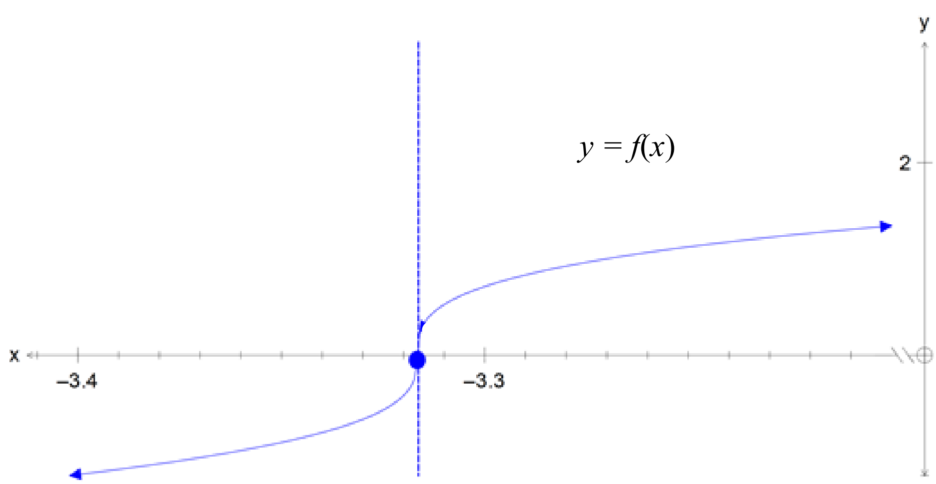
End of Question 13

Question 14 (15 Marks) Use the Question 14 Writing Booklet. **Marks**

a) Prove by mathematical induction that $(3^{2n} - 1)$ is divisible by 8, for all integers $n = 1, 2, 3, \ldots$ **3**

b) A point $P(x, y)$ moves so that its distance from $A(8, -2)$ is equal to twice its distance from $B(-1, 4)$. Find its locus in algebraic form and describe the locus geometrically. **3**

c) The curve $f(x) = (x^3 - 12x)^{\frac{1}{3}}$ is shown below.



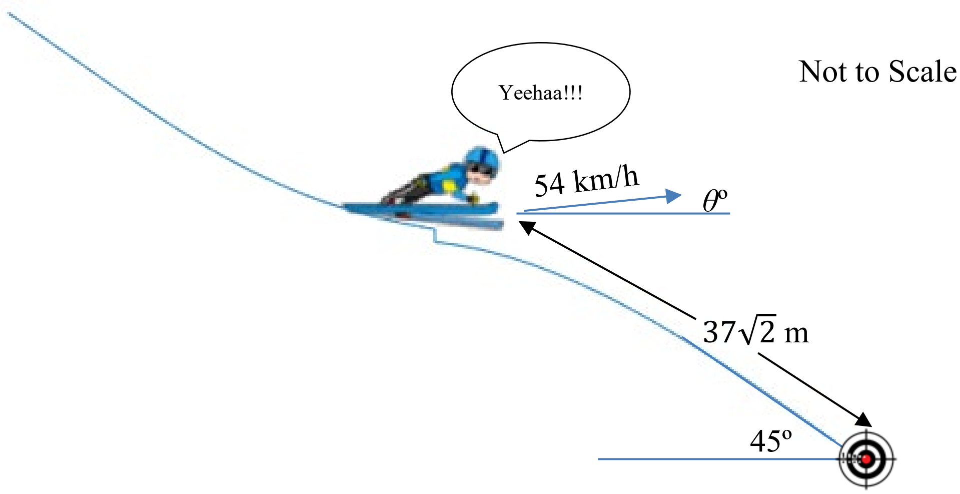
i) It can be seen that $y = f(x)$ crosses the x-axis near $x = -3{\cdot}3$. Use one application of Newton's method to obtain another approximation to the root of $f(x) = 0$. **2**

ii) Explain why using $x = -3{\cdot}3$ as a first approximation does not produce a better approximation to the root than the original approximation. **1**

Question 14 continued on next page

Question 14 (continued)

d) Mr Laurendet has taken some time off work to organize a ski-jump training session for the upcoming snow excursion. He has calculated that if he skis down a slope at Perisher at 54 km/h and launches of a jump he can land on a target $37\sqrt{2}$ metres down the slope. It is known that the slope below the jump falls away at an average rate of 45°.



Let acceleration due to gravity to be $g = -10$ ms^{-2} and Mr Laurendet's angle of projection above the horizontal to be θ,

i) Show that his trajectory path is given by the equation **3**

$$y = -\frac{x^2 \sec^2 \theta}{45} + x \tan \theta.$$

ii) Hence, find the smallest positive angle of projection, θ, to the nearest whole degree, that enables him to land on his target $37\sqrt{2}$ m away. **3**

End of Question 14

End of Examination

2019 MATHEMATICS EXT 1 SOLUTIONS

1) C 2) A 3) D 4) B 5) C
6) A 7) B 8) D 9) D 10) B

Question 11

a) $m_1 = \frac{1}{5}$, $m_2 = \frac{1}{2}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{\frac{1}{5} - \frac{1}{2}}{1 + \frac{1}{5}\times\frac{1}{2}}\right|$

$= \frac{3}{11}$

$\theta \doteqdot 15°$

b) critical pts $x \neq -1$ $3x - 2 = 5(x+1)$
$x = -\frac{7}{2}$

When $x = -2$ LHS $= 8$, True
$x = 0$ LHS $= -2$, False
$x = -10$ LHS $= \frac{-32}{9}$, False

$\therefore -\frac{7}{2} \leq x < -1$

c) i) $\cos 2x = 1 - 2\sin^2 x$

ii) $\lim_{x \to 0} \frac{1 - 2\sin^2 x - 1}{x \sin x} = \lim_{x \to 0} \frac{-2\sin x}{x}$
$= -2$

d) $u = 9 - x^2$
$du = -2x\,dx$
$x = 0$ $u = 9$
$x = 3$ $u = 0$

$\int_0^3 x\sqrt{9 - x^2}\,dx = -\frac{1}{2}\int_9^0 u^{\frac{1}{2}}\,du$
$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^9$
$= \frac{1}{3}(27 - 0)$
$= 9$

e) $\frac{mx_2 - nx_1}{m - n} = \frac{2\times1 + 5\times2}{2 - 5}$ $\frac{my_2 - ny_1}{m - n} = \frac{10 + 5}{-3}$
$= -4$ $= -5$

$P(-4, -5)$

f) i) $T = 35 + Ae^{kt}$ $Ae^{kt} = T - 35$

$\frac{dT}{dt} = kAe^{kt}$
$= k(T - 35)$ as required.

ii) $t = 0$ $5 = 35 + Ae^0$ $\therefore A = -30$
$t = 10$ $10 = 35 - 30e^{10k}$
$e^{10k} = \frac{25}{30}$
$k = \frac{\ln\frac{5}{6}}{10}$

$T = 35 - 30e^{\frac{t}{10}\ln\left(\frac{5}{6}\right)}$ $t = 20$ $T = 14°$

Question 12

a) i) $P(-1) = -1 + b - c - 10 = 0$ $P(2) = 8 + 4b + 2c - 10 = 0$
$b - c = 11$ — ① $4b + 2c = 2$
$2b + c = 1$ — ②

① ② $\Rightarrow 3b = 12$, $b = 4$, $c = -7$

ii) $-1 \times 2 \times \alpha = 10$ or $-1 + 2 + \alpha = -4$
$\alpha = -5$ $\alpha = -5$

b) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{-6}{2} = -3$ $\alpha\beta\gamma = -\frac{1}{2}$ $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 6$

c) let $\tan^{-1}\frac{1}{2} = \alpha$ and $\tan^{-1}\frac{1}{3} = \beta$

$\tan\left[\tan^{-1} x\right] = \tan\left[\alpha - \beta\right]$

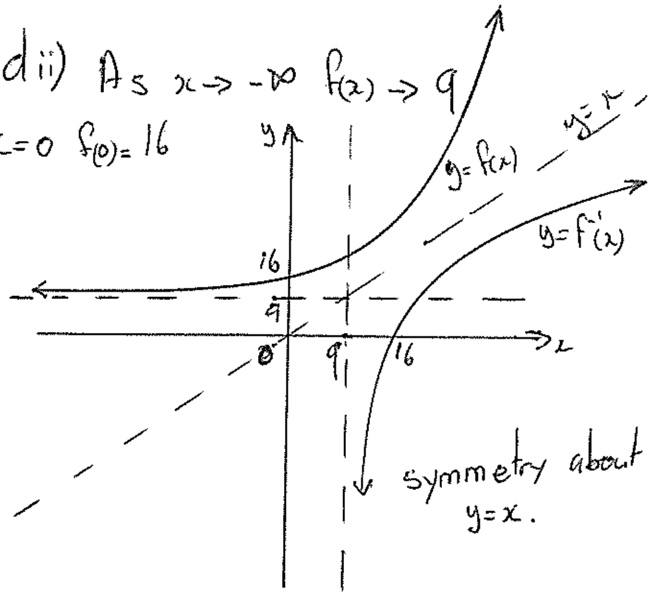
$x = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$= \frac{\frac{1}{2} - \frac{1}{3}}{1 + \frac{1}{2}\times\frac{1}{3}} = \frac{1}{7}$

d) i) $y = e^{2x}$ and $y = e^x$ are monotonic increasing functions, so $f(x)$ is a monotonic increasing function for all x. Hence, a horizontal line will cross only once, so $f^{-1}(x)$ exists for all x.

Q12 d ii) As $x \to -\infty$ $f(x) \to 9$

$x = 0$ $f(0) = 16$



iv)

$x = e^{2y} + 6e^{y} + 9$

let $m = e^{y}$ $\quad e^{2y} = m^2$

$x = m^2 + 6m + 9$

$= (m+3)^2$

$m + 3 = \pm\sqrt{x}$ as $m > 0$ $m + 3 = \sqrt{x}$

$e^{y} + 3 = \sqrt{x}$

$e^{y} = \sqrt{x} - 3$

$y = \ln(\sqrt{x} - 3)$

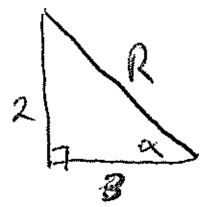
c) i) $3\sin x - 2\cos x$

$R\sin x\cos\alpha - R\cos x\sin\alpha$

$\cos\alpha = \frac{3}{R}$ $\quad \sin\alpha = \frac{2}{R}$

$R = \sqrt{2^2 + 3^2}$

$= \sqrt{13}$ $\quad \tan\alpha = \frac{2}{3}$



$3\sin x - 2\cos x = \sqrt{13}\sin(x - \tan^{-1}\frac{2}{3})$

$\alpha \doteq 0.588$

ii) $\sqrt{13}\sin(x - 0.588) = 1$

$\sin(x - 0.588) = \frac{1}{\sqrt{13}}$

$x - 0.588 = 0.281$

$x \doteq 0.869$ (3 sig fig)

d) i) $m_{OP} \times m_{OQ} = -1$

$m_{OP} = \frac{ap^2}{2ap}$ $\quad m_{OQ} = \frac{aq^2}{2aq}$

$= \frac{p}{2}$ $\quad = \frac{q}{2}$

$\therefore \frac{p}{2} \times \frac{q}{2} = -1$

$pq = -4$ as required

ii) $O(0,0) \to Q$

add $2aq$ to x coord. $(0 + 2aq)$

add aq^2 to y coord. $(0 + aq^2)$

$P(2ap, ap^2) \to R$

add $2aq$ to x coord $(2ap + 2aq)$

add aq^2 to y coord $(ap^2 + aq^2)$

$\therefore R[2a(p+q), a(p^2+q^2)]$

$x = 2a(p+q)$ $\quad y = a(p^2+q^2)$

$p + q = \frac{x}{2a}$ $\quad p^2 + q^2 = \frac{y}{a}$

$(p+q)^2 - 2pq = \frac{y}{a}$

$\left(\frac{x}{2a}\right)^2 - 2\times(-4) = \frac{y}{a}$

$\frac{x^2}{4a^2} + 8 = \frac{y}{a}$

$x^2 = 4ay - 32a^2$

Question 13

a) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 2x\,dx = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos 4x + 1\,dx$

$= \frac{1}{2}\left[\frac{\sin 4x}{4} + x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \frac{1}{8}\left[\sin 2\pi - \sin\frac{4\pi}{3}\right] + \frac{1}{2}\left[\frac{\pi}{2} - \frac{\pi}{3}\right]$

$= \frac{\sqrt{3}}{16} + \frac{\pi}{12}$

b) $\sin\theta = \frac{x}{3}$

$\cos\theta = \frac{y}{4}$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{x^2}{9} + \frac{y^2}{16} = 1$

Q13 a i) $v^2 = 32 + 8x - 4x^2$

$\frac{1}{2}v^2 = 16 + 4x - 2x^2$

$\frac{d(\frac{1}{2}v^2)}{dx} = 4 - 4x$

$= -4(x-1)$

ii) $v = 0$ $\quad 4x^2 - 8x - 32 = 0$

$2x^2 - 4x - 16 = 0$

$(2x-8)(x+2) = 0$

$x = -2$ or $x = 4$

∴ oscillates between $x = -2, 4$

ie amplitude = 3

$T = \frac{2\pi}{n}$ $\quad n^2 = 4$ ∴ $n = 2$

$= \pi$ seconds

iii) Max speed at centre of oscillation

$x = 1$ $\quad v^2 = 32 + 8 - 4$

$= 36$

∴ max speed = 6 m/s

Q14

$n = 1$ $\quad 3^{2n} - 1 = 8$ ∴ true for $n = 1$

Assume true for $n = k$

$3^{2k} - 1 = 8M$ (M is integer-valued)

Prove true for $n = k+1$

$3^{2(k+1)} - 1 = 3^2 \cdot 3^{2k} - 1$

$= 9(8M+1) - 1$

$= 72M + 8$

$= 8(9M+1)$ ∴ divisible by 8

∴ If true for $n=k$, then true for $n=k+1$. As it is true for $n=1$, then by the principle of mathematical induction, it is true for all integers $n = 1, 2, 3, \ldots$

b) $AP = 2BP$

$AP^2 = 4BP^2$

$(x-8)^2 + (y+2)^2 = 4[(x+1)^2 + (y-4)^2]$

$x^2 - 16x + 64 + y^2 + 4y + 4 = 4x^2 + 8x + 4 + 4y^2 - 32y + 64$

$3x^2 + 24x + 3y^2 - 36y = 0$

$x^2 + 8x + y^2 - 12y = 0$

$(x+4)^2 + (y-6)^2 = 52$

circle centre $(-4, 6)$ radius $= \sqrt{52}$

$= 2\sqrt{13}$

c) i) $f(x) = (x^3 - 12x)^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}(3x^2 - 12)(x^3 - 12x)^{-\frac{2}{3}}$

$= \frac{x^2 - 4}{(x^3 - 12x)^{\frac{2}{3}}}$

$x_1 = -3.3$ $\quad f(-3.3) \doteq 1.542$

$f'(-3.3) \doteq 2.900$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= -3.3 - \frac{1.542}{2.9}$

$= -3.83$ (3 sig fig)

ii) Newton's method uses the x-intercepts of tangents to find an approximate. At $x = -3.3$ the slope of $y = f(x)$ is not very steep and this pushes the tangent away from the root rather than closer.

Q14d

i) $\dot{x} = V\cos\alpha \qquad \dot{y} = -gt + V\sin\alpha$

$V = 54\text{ km/h} = 15\text{ ms}^{-1}$

$\dot{x} = 15\cos\theta \qquad \dot{y} = -10t + 15\sin\theta$

$x = 15t\cos\theta + c \qquad y = -5t^2 + 15t\sin\theta + c$

$t = 0 \quad x = 0 \quad y = 0 \quad \therefore$ Both $c = 0$.

$x = 15t\cos\theta \qquad y = -5t^2 + 15t\sin\theta$

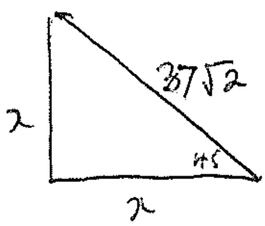
$$t = \frac{x}{15\cos\theta}$$

$$y = -5\left(\frac{x}{15\cos\theta}\right)^2 + 15\left(\frac{x}{15\cos\theta}\right)\sin\theta$$

$$= \frac{-x^2}{45\cos^2\theta} + \frac{x\sin\theta}{\cos\theta}$$

$$= -\frac{x^2\sec^2\theta}{45} + x\tan\theta.$$

ii) $x = 37 \qquad y = -37$



$x^2 + x^2 = (37\sqrt{2})^2$

$2x^2 = 37^2 \times 2$

$x = 37,\ y = 37$

$$-37 = -\frac{(37^2)\sec^2\theta}{45} + 37\tan\theta$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$-37 = -\frac{37^2}{45}(1 + \tan^2\theta) + 37\tan\theta$$

$$-45 = -37 - 37\tan^2\theta + 45\tan\theta$$

$$37\tan^2\theta - 45\tan\theta - 8 = 0$$

let $m = \tan\theta$

$$37m^2 - 45m - 8 = 0$$

$$m = \frac{45 \pm \sqrt{45^2 + 4 \times 37 \times 8}}{74}$$

$$= 1.373 \text{ or } -0.157$$

as $m > 0$

$$\theta = \tan^{-1}(1.373)$$

$$= 54^\circ$$